5.4 DERIVATIVE RANDOM PROCESSES

The derivative of any particular sample function $x(t)$ from an arbitrary random process $\{x(t)\}$ is defined by

$$\dot{x}(t) = \frac{dx(t)}{dt} = \lim_{\varepsilon \to 0} \left[\frac{x(t+\varepsilon) - x(t)}{\varepsilon}\right] \tag{5.144}$$

Existence of this limit may occur in different ways. The derivative $\dot{x}(t)$ is said to exist

1. In the *usual sense* if the limit exists for all functions $x(t)$ in $\{x(t)\}$.
2. In the *mean square sense* if

$$\lim_{\varepsilon \to 0} E\left[\left|\frac{x(t+\varepsilon) - x(t)}{\varepsilon} - \dot{x}(t)\right|^2\right] = 0 \tag{5.145}$$

For a stationary random process, a necessary and sufficient condition for $\dot{x}(t)$ to exist in the mean square sense is that its autocorrelation function $R_{xx}(\tau)$ should have derivatives of order up to 2; that is, $R'_{xx}(\tau)$ and R''_{xx} must exist [Ref. 5].

5.4.1 Correlation Functions

Consider the following derivative functions, which are assumed to be well defined:

$$\begin{aligned} R'_{xx}(\tau) &= \frac{dR_{xx}(\tau)}{d\tau} & R''_{xx}(\tau) &= \frac{d^2R_{xx}(\tau)}{d\tau^2} \\ \dot{x}(t) &= \frac{dx(t)}{dt} & \ddot{x}(t) &= \frac{d^2x(t)}{dt^2} \end{aligned} \tag{5.146}$$

By definition, for stationary random data,

$$\begin{aligned} R_{xx}(\tau) &= E[x(t)x(t+\tau)] = E[x(t-\tau)x(t)] \\ R_{x\dot{x}}(\tau) &= E[x(t)\dot{x}(t+\tau)] = E[x(t-\tau)\dot{x}(t)] \\ R_{\dot{x}\dot{x}}(\tau) &= E[\dot{x}(t)\dot{x}(t+\tau)] = E[\dot{x}(t-\tau)\dot{x}(t)] \end{aligned} \tag{5.147}$$

Now

$$R'_{xx}(\tau) = \frac{d}{d\tau}E[x(t)x(t+\tau)] = E[x(t)\dot{x}(t+\tau)] = R_{x\dot{x}}(\tau) \tag{5.148}$$

Also,

$$R'_{xx}(\tau) = \frac{d}{d\tau}E[x(t-\tau)x(t)] = -E[\dot{x}(t-\tau)x(t)] = -R_{\dot{x}x}(\tau)$$

Hence,

$$R'_{xx}(0) = R_{x\dot{x}}(0) = -R_{\dot{x}x}(0) = 0 \tag{5.149}$$

since $R'_{xx}(0)$ equals the positive and negative of the same quantity. The corresponding $R_{xx}(0)$ is a maximum value of $R_{xx}(\tau)$. This proves that for stationary random data

$$E[x(t)\dot{x}(t)] = 0 \tag{5.150}$$

In words, at any t, Equation (5.150) indicates that the derivative $\{\dot{x}(t)\}$ for stationary random data $\{x(t)\}$ is equally likely to be positive or negative. Equation (5.148) states that the derivative $R'_{xx}(\tau)$ of the autocorrelation function $R_{xx}(\tau)$ with respect to τ is the same as the cross-correlation function between $\{x(t)\}$ and $\{\dot{x}(t)\}$. A maximum value for the autocorrelation function $R_{xx}(\tau)$ corresponds to a zero crossing for its derivative $R'_{xx}(t)$, which becomes a zero crossing for the cross-correlation function between $\{x(t)\}$ and $\{\dot{x}(t)\}$. This crossing of zero by $R'_{xx}(\tau)$ will be with *negative* slope, that is

$$R'_{xx}(0-) > 0 \quad \text{and} \quad R'_{xx}(0+) < 0 \tag{5.151}$$

as can be seen from the picture in Figure 5.9. In practice, determining the location where zero crossings will occur is usually easier than determining the location of maximum values.

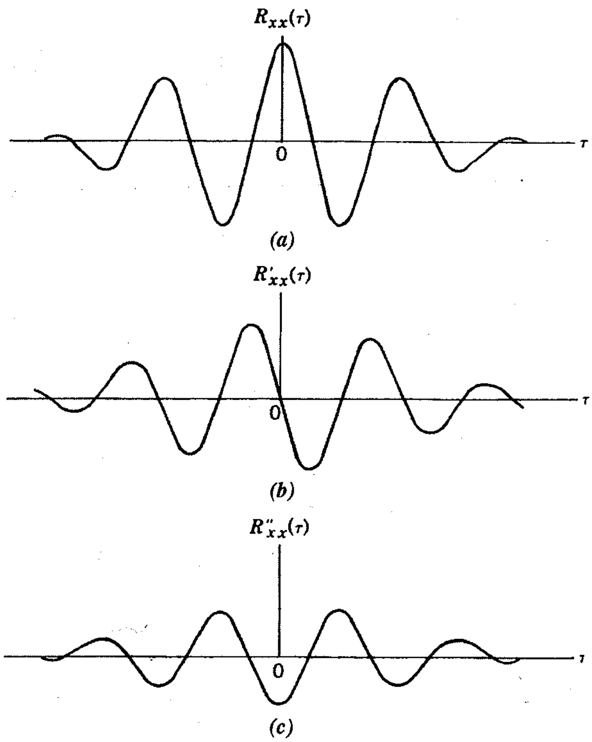

Figure 5.9 Illustration of derivatives of autocorrelation functions. (*a*) Original function. (*b*) First derivative. (*c*) Second derivative.

It will now be shown that $R'_{xx}(\tau)$ is an odd function of τ corresponding to $R_{xx}(\tau)$ being an even function of τ. By definition

$$R_{xx}(-\tau) = E[x(t)x(t-\tau)] = E[x(t+\tau)x(t)] \tag{5.152}$$

Hence,

$$R'_{xx}(-\tau) = \frac{d}{d\tau}E[x(t+\tau)x(t)] = E[\dot{x}(t+\tau)x(t)] = R_{\dot{x}x}(\tau) \tag{5.153}$$

But, as shown earlier, $R_{\dot{x}x}(\tau) = -R'_{xx}(\tau)$. Hence Equation (5.153) becomes

$$R'_{xx}(-\tau) = -R'_{xx}(\tau) \tag{5.154}$$

This proves that $R'_{xx}(\tau)$ is an odd function of τ.

The second derivative gives

$$\begin{aligned} R''_{xx}(\tau) &= \frac{d}{d\tau}R'_{xx}(\tau) = \frac{d}{d\tau}R_{x\dot{x}}(\tau) = \frac{d}{d\tau}E[x(t-\tau)\dot{x}(t)] \\ &= -E[\dot{x}(t-\tau)\dot{x}(t)] = -R_{\dot{x}\dot{x}}(\tau) \end{aligned} \tag{5.155}$$

Also,

$$\begin{aligned} R''_{xx}(\tau) &= \frac{d}{d\tau}R'_{xx}(\tau) = \frac{d}{d\tau}R_{x}(\tau) = \frac{d}{d\tau}E[x(t)\dot{x}(t+\tau)] \\ &= E[x(t)\ddot{x}(t+\tau)] = R_{x\ddot{x}}(\tau) \end{aligned} \tag{5.156}$$

One can also verify directly that $R''_{xx}(\tau)$ is an even function of τ, namely,

$$R''_{xx}(-\tau) = R''_{xx}(\tau) \tag{5.157}$$

At $\tau = 0$, one obtains

$$E[\dot{x}^2(t)] = R_{\dot{x}\dot{x}}(0) = -R_{x\ddot{x}}(0) - R''_{xx}(0) \tag{5.158}$$

As shown earlier,

$$R_{x\dot{x}}(\tau) = \frac{d}{d\tau}R_{xx}(\tau) = R'_{xx}(\tau) \tag{5.159}$$

Typical plots for $R_{xx}(\tau)$, $R'_{xx}(\tau)$, and $R''_{xx}(\tau)$ are drawn in Figure 5.9, based on a sine wave process where

$$\begin{aligned} R_{xx}(\tau) &= X \cos 2\pi f_0 \tau \\ R'_{xx}(\tau) &= -X(2\pi f_0)\sin 2\pi f_0 \tau \\ R''_{xx}(\tau) &= -X(2\pi f_0)^2 \cos 2\pi f_0 \tau \end{aligned} \tag{5.160}$$

The results given above can be extended to higher order derivatives. For example,

$$R_{\dot{x}\ddot{x}}(\tau) = \frac{d}{d\tau}R_{\dot{x}\dot{x}}(\tau) = -R'''_{xx}(\tau) \tag{5.161}$$

$$R_{\ddot{x}\ddot{x}}(\tau) = -\frac{d}{d\tau}R_{\dot{x}\ddot{x}}(\tau) = R''''_{xx}(\tau) \tag{5.162}$$

At $\tau = 0$, one obtains

$$E[\ddot{x}^2(t)] = R_{\ddot{x}\ddot{x}}(0) = R''''_{xx}(0) \tag{5.163}$$

Thus, knowledge of $R_{xx}(\tau)$ and its successive derivatives can enable one to state the properties for autocorrelation and cross-correlation functions between $\{x(t)\}$ and its successive derivatives $\{\dot{x}(t)\}$, $\{\ddot{x}(t)\}$, and so on.

5.4.2 Spectral Density Functions

It is easy to derive corresponding properties for the autospectral and cross-spectral density functions between $\{x(t)\}$ and its successive derivatives $\{\dot{x}(t)\}$ and $\{\ddot{x}(t)\}$. Let

$$X(f) = \mathscr{F}[x(t)] = \text{Fourier transform}[x(t)] \tag{5.164}$$

Then

$$\mathscr{F}[\dot{x}(t)] = (j2\pi f)X(f) \tag{5.165}$$

$$\mathscr{F}[\ddot{x}(t)] = -(2\pi f)^2 X(f) \tag{5.166}$$

From Equations (5.66) and (5.67), it follows directly that

$$G_{x\dot{x}}(f) = j(2\pi f)G_{xx}(f) \tag{5.167}$$

$$G_{\dot{x}\dot{x}}(f) = (2\pi f)^2 G_{xx}(f) \tag{5.168}$$

$$G_{\dot{x}\ddot{x}}(f) = j(2\pi f)^3 G_{xx}(f) \tag{5.169}$$

$$G_{\ddot{x}\ddot{x}}(f) = (2\pi f)^4 G_{xx}(f) \tag{5.170}$$

and so on. These formulas are the same with one-sided G's replaced by the corresponding two-sided S's.

These results can also be derived from the Wiener–Khincnine relations of Equation (5.28). Start with the basic relation

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f)e^{j2\pi f\tau}df \tag{5.171}$$

Then successive derivatives will be

$$R'_{xx}(\tau) = j\int_{-\infty}^{\infty} (2\pi f)S_{xx}(f)e^{j2\pi f\tau}df \tag{5.172}$$

$$R''_{xx}(\tau) = -\int_{-\infty}^{\infty} (2\pi f)^2 S_{xx}(f)e^{j2\pi f\tau}df \tag{5.173}$$

$$R_{xx}'''(\tau) = -j\int_{-\infty}^{\infty} (2\pi f)^3 S_{xx}(f) e^{j2\pi f\tau} df \tag{5.174}$$

$$R_{xx}''''(\tau) = \int_{-\infty}^{\infty} (2\pi f)^4 S_{xx}(f) e^{j2\pi f\tau} df \tag{5.175}$$

The Wiener-Khinchine relations, together with previous formulas in Section 5.4.1, show that these four derivative expressions are the same as

$$R_{xx}'(\tau) = R_{x\dot{x}}(\tau) = \int_{-\infty}^{\infty} S_{x\dot{x}}(f) e^{j2\pi f\tau} df \tag{5.176}$$

$$R_{xx}''(\tau) = -R_{\dot{x}\dot{x}}(\tau) = -\int_{-\infty}^{\infty} S_{\dot{x}\dot{x}}(f) e^{j2\pi f\tau} df \tag{5.177}$$

$$R_{xx}'''(\tau) = -R_{\dot{x}\ddot{x}}(\tau) = -\int_{-\infty}^{\infty} S_{\dot{x}\ddot{x}}(f) e^{j2\pi f\tau} df \tag{5.178}$$

$$R_{xx}''''(\tau) = R_{\ddot{x}\ddot{x}}(\tau) = \int_{-\infty}^{\infty} S_{\ddot{x}\ddot{x}}(f) e^{j2\pi f\tau} df \tag{5.179}$$

Corresponding terms in the last eight formulas yield Equations (5.167)–(5.170).

5.5 LEVEL CROSSINGS AND PEAK VALUES

This section addresses the probability functions for certain characteristics of random data including

1. Expected number of level crossings per unit time
2. Peak probability functions for narrow bandwidth data
3. Expected number and spacing of positive peaks
4. Peak probability functions for wide bandwidth data

Most of these results were originally derived by Rice in Ref. 6, and various extensions of these matters are presented in Refs 6–10.

5.5.1 Expected Number of Level Crossings per Unit Time

Consider a stationary random record $x(t)$ that has the time derivative $v(t) = \dot{x}(t)$. Let $p(\alpha, \beta)$ represent the joint probability density function of $x(t)$ and $v(t)$ at $x(t) = \alpha$ and $v(t) = \beta$. By definition, for all t,

$$p(\alpha,\beta)\Delta\alpha\Delta\beta \approx \text{Prob}[\alpha < x(t) \leq \alpha + \Delta\alpha \text{ and } \beta < v(t) \leq \beta + \Delta\beta] \tag{5.180}$$

In words, $p(\alpha, \beta)\,\Delta\alpha\Delta\beta$ estimates the probability over all time that $x(t)$ lies in the interval $[\alpha, \alpha + \Delta\alpha]$ when its derivative $v(t)$ is in the interval $[\beta, \beta + \Delta\beta]$. For unit total time, when $\Delta\beta$ is negligible compared to β, the value of $v(t)$ is essentially β.

To find the expected number of crossings of $x(t)$ through the interval $[\alpha, \alpha + \Delta\alpha]$, the amount of time that $x(t)$ is inside the interval should be divided by the time required to cross the interval. If t_β is the crossing time for a particular derivative β, then

$$t_\beta = \frac{\Delta\alpha}{|\beta|} \tag{5.181}$$

where the absolute value of β is used because the crossing time must be a positive quantity. Hence, the expected number of passages of $x(t)$ per unit time through the interval $[\alpha, \alpha + \Delta\alpha]$ for a given value of $v(t) = \beta$ is

$$\frac{p(\alpha, \beta)\Delta\alpha\Delta\beta}{t_\beta} = |\beta| p(\alpha, \beta)\Delta\beta \tag{5.182}$$

In the limit as $\Delta\beta \to 0$, the total expected number of crossings of $x(t)$ per unit time through the level $x(t) = \alpha$ for all possible values of β is found by

$$N_\alpha = \int_{-\infty}^{\infty} |\beta| p(\alpha, \beta) d\beta \tag{5.183}$$

This represents the expected number of crossings of the level α with both positive and negative slopes as shown in Figure 5.10.

The expected number of zeros of $x(t)$ per unit time is found by the number of crossings of the level $x(t) = 0$ with both positive and negative slopes. This is given by N_α when $\alpha = 0$, namely,

$$N_0 = \int_{-\infty}^{\infty} |\beta| p(0, \beta) d\beta \tag{5.184}$$

This value of N_0 can be interpreted as twice the "apparent frequency" of the record. For example, if the record were a sine wave of frequency f_0 Hz, then N_0 would be $2f_0$ zeros per second (e.g., a 60 Hz sine wave has 120 zeros per second). For random data, the situation is more complicated, but still a knowledge of N_0, together with other quantities, helps to characterize the particular random data. The above formulas apply to arbitrary Gaussian or non-Gaussian random data.

For an arbitrary random record $x(t)$ and its derivative $v(t) = \dot{x}(t)$ from a zero mean value stationary random process, it follows from Equations (5.147), (5.150) and (5.158)

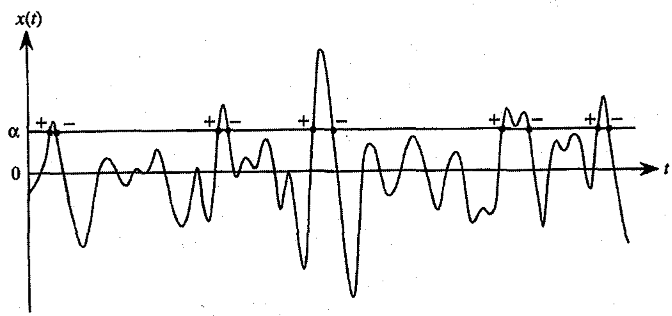

Figure 5.10 Illustration of crossings of level α with positive and negative slopes.

that the variances and covariance are

$$\sigma_x^2 = E[x^2(t)] = R_{xx}(0) \tag{5.185}$$

$$\sigma_v^2 = E[v^2(t)] = R_{vv}(0) = -R''_{xx}(0) \tag{5.186}$$

$$\sigma_{xv} = E[x(t)v(t)] = 0 \tag{5.187}$$

From Equations (5.171) and (5.173), it also follows that

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_{xx}(f)df = \int_0^{\infty} G_{xx}(f)df \tag{5.188}$$

$$\sigma_v^2 = \int_{-\infty}^{\infty} (2\pi f)^2 S_{xx}(f)df = \int_0^{\infty} (2\pi f)^2 G_{xx}(f)df \tag{5.189}$$

If $x(t)$ and $v(t)$ are statistically independent with $p(\alpha, \beta) = p(\alpha)q(\beta)$, then Equations (5.183) and (5.184) show that

$$\frac{N_\alpha}{N_0} = \frac{p(\alpha)}{p(0)} \tag{5.190}$$

regardless of the nature of $p(\alpha)$ or $q(\beta)$. The formulas in Equations (5.18)–(5.19) apply to stationary random data with any joint probability density function.

5.5.1.1 *Gaussian Data*

Assume now that $x(t)$ and its derivative $v(t) = \dot{x}(t)$ have zero mean values and form a joint Gaussian distribution with the above variances and zero covariance. Then, the joint probability density function

$$p(\alpha, \beta) = p(\alpha)q(\beta) \tag{5.191}$$

with

$$p(\alpha) = \left(\frac{1}{\sigma_x\sqrt{2\pi}}\right)\exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right) \tag{5.192}$$

$$q(\beta) = \left(\frac{1}{\sigma_v\sqrt{2\pi}}\right)\exp\left(-\frac{\beta^2}{2\sigma_v^2}\right) \tag{5.193}$$

Substitution of Equation (5.190) into (5.183) now shows for Gaussian data that

$$N_\alpha = \frac{\exp(-\alpha^2/2\sigma_x^2)}{2\pi\sigma_x\sigma_v}\int_{-\infty}^{\infty} |\beta|\exp\left(-\frac{\beta^2}{2\sigma_v^2}\right)d\beta = \frac{1}{\pi}\left(\frac{\sigma_v}{\sigma_x}\right)\exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right) \tag{5.194}$$

In particular, for $\alpha = 0$, one obtains the simple formula

$$N_0 = \frac{1}{\pi}\left(\frac{\sigma_v}{\sigma_x}\right) \tag{5.195}$$