satisfied. Result II then follows from Result I. In practice, these conditions are usually satisfied, justifying the assumption of ergodicity.

5.3.3 Gaussian Random Processes

The formal definition of a Gaussian random process is as follows. A random process $\{x_k(t)\}$ is said to be a Gaussian random process if, for every set of fixed times $\{t_n\}$, the random variables $x_k(t_n)$ follow a multidimensional normal distribution as defined by Equation (3.63). Gaussian random processes are quite prevalent in physical problems and often may be mathematically predicted by the multidimensional central limit theorem. Also, it can be shown that if a Gaussian process undergoes a linear transformation, then the output will still be a Gaussian process. This property is quite important in various theoretical and practical applications of random process theory.

Consider a time history $x(t)$, which is a sample function from an ergodic Gaussian random process with a zero mean value. Note that the index k is no longer needed because the properties of any one sample function will be representative of all other sample functions. From the ergodic property, the behavior of $x(t)$ over a long period of time will exhibit the same statistical characteristics as corresponding ensemble averages at various fixed times. As a consequence, it follows that the probability density function associated with the instantaneous values of $x(t)$ that will occur over a long time interval is given by the Gaussian probability density function with zero mean value, as follows:

$$p(x) = (\sigma_x\sqrt{2\pi})^{-1}e^{-x^2/2\sigma_x^2} \tag{5.124}$$

The variance σ_x^2 when $x(t)$ has a zero mean is determined by

$$\begin{aligned} \sigma_x^2 = E[x^2(t)] &= \int_{-\infty}^{\infty} x^2p(x)dx \quad \text{independent of } t \\ &\approx \frac{1}{T}\int_0^T x^2(t)dt \quad \text{for large } T \\ &= \int_{-\infty}^{\infty} S_{xx}(f)df = 2\int_0^{\infty} S_{xx}(f)df = \int_0^{\infty} G_{xx}(f)df \end{aligned} \tag{5.125}$$

Thus, the Gaussian probability density function $p(x)$ is completely characterized through a knowledge of $S_{xx}(f)$ or $G_{xx}(f)$ since they alone determine σ_x. This important result places a knowledge of $S_{xx}(f)$ or $G_{xx}(f)$ at the forefront of much work in the analysis of random records. It should be noted that no restriction is placed on the shape of the autospectral density function or its associated autocorrelation function.

If the mean value of $x(t)$ is not zero, then the underlying probability density function is given by the general Gaussian formula

$$p(x) = (\sigma_x\sqrt{2\pi})^{-1}e^{-(x-\mu_x)^2/2\sigma_x^2} \tag{5.126}$$

where the mean value

$$\begin{aligned} \mu_x = E[x(t)] &= \int_{-\infty}^{\infty} xp(x)dx \quad \text{independent of } t \\ &\approx \frac{1}{T}\int_0^T x(t)dt \quad \text{for large } T \end{aligned} \tag{5.127}$$

and the variance

$$\sigma_x^2 = E[(x(t)-\mu_x)^2] = E[x^2(t)]-\mu_x^2 \tag{5.128}$$

Assume that $\{x(t)\}$ is a stationary Gaussian random process where the index k is omitted for simplicity in notation. Consider the two random variables $x_1 = x(t)$ and $x_2 = x(t+\tau)$ at an arbitrary pair of fixed times t and $t+\tau$. Assume that x_1 and x_2 follow a two-dimensional (joint) Gaussian distribution with *zero means* and *equal variances* σ_x^2. By definition, then

$$\sigma_x^2 = E[x^2(t)] = E[x^2(t+\tau)] = \int_{-\infty}^{\infty} x^2 p(x)dx \tag{5.129}$$

$$R_{xx}(\tau) = E[x(t)x(t+\tau)] = \rho_{xx}(\tau)\sigma_x^2 = \iint_{-\infty}^{\infty} x_1 x_2 p(x_1, x_2)dx_1 dx_2 \tag{5.130}$$

The quantity $\rho_{xx}(\tau)$ is the correlation coefficient function of Equation (5.16) for $C_{x_1x_2}(\tau) = R_{xx}(\tau)$ and $\sigma_{x_1} = \sigma_{x_2} = \sigma_x$, namely,

$$\rho_{xx}(\tau) = \frac{R_{xx}(\tau)}{\sigma_x^2} \tag{5.131}$$

Letting $\rho = \rho_{xx}(\tau)$ and $\mu = 0$, the joint Gaussian probability density function is given by

$$p(x_1, x_2) = (2\pi\sigma_x^2\sqrt{1-\rho^2})^{-1}\exp\left[\frac{-1}{2\sigma_x^2(1-\rho^2)}(x_1^2 - 2\rho x_1 x_2 + x_2^2)\right] \tag{5.132}$$

All properties developed in Section 3.3 apply to joint Gaussian random processes at any set of fixed times.

Consider four random variables x_1, x_2, x_3, x_4, with zero mean values, which follow a four-dimensional Gaussian distribution. From Equation (3.73),

$$E[x_1x_2x_3x_4] = E[x_1x_2]E[x_3x_4] + E[x_1x_3]E[x_2x_4] + E[x_1x_4]E[x_2x_3] \tag{5.133}$$

In particular, let $x_1 = x(u)$, $x_2 = y(u+\tau)$, $x_3 = x(v)$, $x_4 = y(v+\tau)$, and let $R_{xy}(\tau)$ be the stationary cross-correlation function given by

$$R_{xy}(\tau) = E[x(t)y(t+\tau)] \tag{5.134}$$

It now follows from Equation (5.133) that

$$\begin{aligned} E[x(u)y(u+\tau)x(v)y(v+\tau)] = {} & R_{xy}^2(\tau) + R_{xx}(v-u)R_{yy}(v-u) \\ & + R_{xy}(v-u+\tau)R_{yx}(v-u-\tau) \end{aligned} \tag{5.135}$$

This result will be used later in Equation (8.99).