Self-Checking Resistive Ratios

ARTHUR MELVILLE THOMPSON

Abstract—**The two resistors which form the ratio are constructed as two separate networks of nominally equal resistors. Each network is chosen such that there is a dual network which can be constructed from the same set of resistors. Two configurations with the same nominal ratio are obtained by switching each network to its dual and interchanging their positions in the ratio. The mean ratio is a very close approximation to the nominal value. As an example, it is shown that only six resistors are required to produce all steps of a single-decade voltage divider with a mean accuracy of 1 in 10^6 using resistors adjusted to 1 in 10^3.**

I. Introduction

THERE ARE many measurement systems in which the ratio of two quantities is derived from the ratio of two resistances, and many methods have been devised for calibrating such ratios. The two resistance values may be measured separately on a suitable bridge, in which case the calibration problem is transferred to the bridge. We are concerned with constructing the two resistors in such a way that the resistance ratio may be determined by additional measurements on the same measurement system, i.e., to provide for *in situ* calibration.

The simplest and best known example of such an *in situ* calibration, which is also the simplest case of the general system to be described, is a ratio of unity given by two, nominally equal resistances R_1 and R_2. In practice R_1 and R_2 will not be exactly equal, and the ratio will not be exactly unity. A measurement is made with the ratio $X = R_1/R_2$ so that the value obtained is proportional to X. R_1 and R_2 are interchanged to give a ratio $Y = R_2/R_1$, and another measurement gives a value which is proportional to Y. Since $(XY)^{1/2} = 1$, the geometric mean of the two values will correspond to a ratio of unity. Since $R_1/R_2 = (X/Y)^{1/2}$, this ratio may also be determined from the two measurements.

If $R_1/R_2 = 1 + \delta$, where δ is small, then the arithmetic mean of the two measurements which corresponds to $\frac{1}{2}(X + Y) = 1 + \frac{1}{2}\delta^2, \cdots$ will be a very close approximation to the unity ratio value.

Ratios other than unity may also be made self-checking if the two ratio resistors are each made up of combinations of nominally equal resistors, i.e., each resistance value is the input resistance of a network of nominally equal resistances. For a ratio in which the total number of resistors in the two networks is small, a very accurate value may be obtained by taking the average of all the values corresponding to the different approximations to the ratio obtained by permuting the resistors in the two networks. This method becomes rather time consuming as the number of permutations increases.

It is possible to choose the networks so that only two measurements are required. If each of the two networks of equal resistances has a dual, then two configurations with the same ratio can be obtained by switching each network to its dual and interchanging their positions in the ratio. It will be shown that if the resistances are not quite equal, the mean ratio is a very close approximation to the true value.

II. Dual Networks

We shall be concerned only with passive, resistive networks. If the graph of a network can be mapped on a sphere, then there is a dual network of the same number of branches such that the meshes of one correspond to the nodes of the other. If, in addition, the values of the resistances of the branches of one network are equal to the conductances of the corresponding branches in the other, then all corresponding dual quantities of the two networks are equal. In particular, the input resistance between a pair of terminals of one network is equal to the input conductance between corresponding terminals of the other. If we have a network in which all the branch resistances are equal to R, then in the dual network all the branch conductances are equal to G, and for all the duality relations to hold $R = G$. For $R \neq G$ we have networks with dual geometries only, but we can easily determine the relations between corresponding quantities by using the duality relations for $R = G = 1$ and scaling the results. For example, if the resistance between two terminals of a network of 1-Ω resistances is p, the resistance between the terminals of a similar network with resistances R is pR. The conductance between corresponding terminals of the dual network with 1-S conductances must also be p so that the conductance between the terminals of a similar network with conductances G is pG. We shall use this scaling process for the case where the same components are used in each network, i.e., $RG = 1$.

III. Resistive Ratios

Consider the ratio of two resistances, one derived from a network of resistors each of value R_1 and the other from another network of resistors each of value R_2. If the resistance of the first network is pR_1 and that of the second qR_2, then the resistance ratio is

$$X = pR_1/qR_2.$$

Now if each network is switched to the dual configuration using the same components, the conductances corresponding to the resistance values of the original networks are pG_1

Manuscript received April 24, 1978.

The author is with the National Measurement Laboratory, CSIRO, Sydney, Australia 2070.

0018-9456/78/1200-0423$00.75

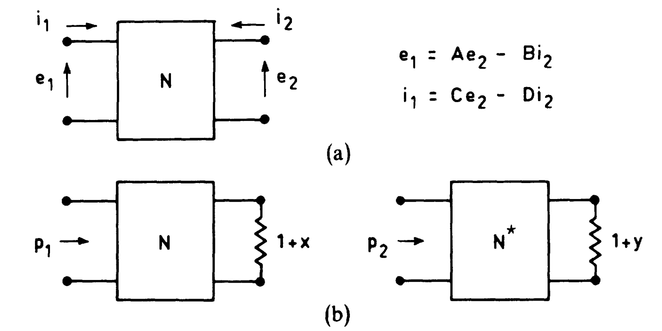

Fig. 1. (a) General circuit parameters for a two-terminal-pair network. (b) Dual configurations for one offset component.

and qG_2 where $G_1 = 1/R_1$ and $G_2 = 1/R_2$. If these networks are interchanged the resistance ratio is

$$Y = pR_2/qR_1.$$

Hence $(XY)^{1/2} = p/q$, i.e., as for the unity ratio considered earlier, the geometric mean gives the required ratio p/q. As before if $R_1/R_2 = 1 + \delta$, $\frac{1}{2}(X + Y) = (1 + \frac{1}{2}\delta^2, \cdots)p/q$.

To evaluate the effect of inequalities amongst the resistances of one of the networks, we return to the $R = G = 1$ networks and assume that one of the resistances is $1 + x$ and the corresponding conductance is $1 + y$, i.e., $(1 + x) \times (1 + y) = 1$. The remaining 1-$\Omega$ resistances form a two-terminal-pair network N which is terminated by the resistance of $1 + x$ and whose input resistance p_1 would equal p for $x = 0$. For the dual configuration the 1-S conductances form a two-terminal-pair network N^* which is the dual of N and which is terminated by the conductance of $1 + y$ and whose input conductance p_2 would equal p for $y = 0$. From Fig. 1, we have $e_2 = -i_2(1 + x)$, whence

$$p_1 = e_1/i_1 = \frac{A(1 + x) + B}{C(1 + x) + D}$$

and from the duality of N and N^*

$$p_2 = \frac{A(1 + y) + B}{C(1 + y) + D}.$$

If the networks are now scaled to R_1, G_1 and we consider the same divider as before, we have $X = p_1 R_1/qR_2$ and $Y = p_2 R_2/qR_1$.

Hence

$$(XY)^{1/2} = (p_1 p_2)^{1/2}/q$$

and $(p_1 p_2)^{1/2}$ replaces the p of our original ratio where

$$p_1 p_2 = \frac{[A(1 + x) + B][A(1 + y) + B]}{[C(1 + x) + D][C(1 + y) + D]} = \frac{(A + B)^2 - ABxy}{(C + D)^2 - CDxy}$$

and

$$(p_1 p_2)^{1/2} = \frac{A+B}{C+D}\left[1 - xy\left(\frac{AB}{(A+B)^2} - \frac{CD}{(C+D)^2}\right), \cdots\right]^{1/2} = p(1 - W \cdot xy, \cdots).$$

TABLE I
RESISTANCE VALUES OBTAINED BY COMBINATIONS OF 1-Ω RESISTORS

Number of 1Ω Resistors	Size of Unit 1	1/2	1/3	1/4	1/5	1/6	1/7	1/8
1	1							
2	2	1						
3	3	3	1 2					
4	1 4	5	4 5	1 3	2 3			
5	2 5	1 7	7 8	5 7	1 4 6 7 8	5 7	2 3 4 5 6	3 5

The number in the Table is to be multiplied by the size of the unit at the head of the column.

The value of W can be calculated for each resistor in the network. Since for resistance networks A, B, C, and D, all have the same sign, $W < \frac{1}{8}$.

We have shown that the offsetting of a single resistor produces only a second-order effect on the mean ratio. For the general case where the errors in the resistors are distributed about a mean, the second-order error contains all product terms with coefficients that depend on the positions of the components in the network. The number of terms increases rapidly with the number of resistors, and, in a particular case, the errors are best evaluated by inserting measured values in the expressions for the input resistance and input conductance of the two configurations. Examples of this approach are given later.

IV. Applications

A special case of the dual configuration technique has been used to provide a very accurate self-checking potential divider to compare the EMF of a standard cell with the potential produced by a biased and irradiated Josephson junction [1]. Two Hamon-type buildup resistors [2] are used to form the ratio. This type of resistor is built-up from a number of equal resistors which may be switched from series to parallel. For n resistors in the buildup, the ratio of series to parallel resistance is very close to n^2. The two configurations of the ratio are obtained by switching each buildup resistor to its alternative configuration and interchanging the resistors in the divider. With buildup resistors of n and m nominally equal resistors, respectively, ratios of n/m or $n \times m$ may be set up, but the system has usually been used with $n = m$ for a ratio of n^2. For this case the input and output resistances of the potential divider are both unchanged by the change of configuration.

To illustrate the usefulness of the dual configuration technique for setting up self-checking resistive ratios, we shall consider its application to the design of a single-decade potential divider, i.e., we wish to realize networks with resistance ratios of 9:1, 8:2, 7:3, 6:4, and 5:5.

The selection of network configurations is aided by Table I which gives the resistance values that can be obtained by

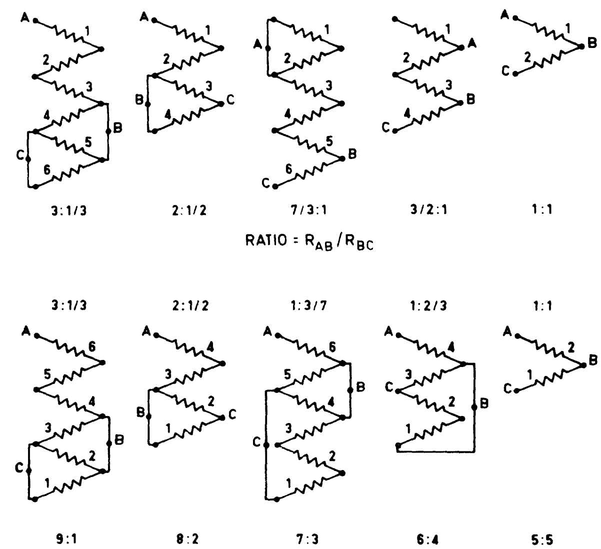

Fig. 2. Dual configurations of networks producing resistance ratios for each step of a single-decade divider.

various combinations of a small number of 1-Ω resistances. The number of combinations with different values of resistance increases very rapidly as the number of resistances in the network is increased.

The required ratios for all the steps of the decade can be obtained from only six resistors, as indicated in Fig. 2, which shows the set of dual configurations. The resistors are numbered as though permanently connected in series. The remainder of the decade is produced by inversion of the ratios shown.

We can calculate the individual ratios and the errors of the means if we know the values of the resistances. As an example, we shall assume that all of the odd-numbered resistors are high by 1 in 10^3 and all of the even-numbered low by the same amount. The individual ratios and the means are given in Table II. It can be seen that for this by no means favorable distribution, the geometric mean is in error by not more than 1 ppm and the arithmetic mean by not more than 2 ppm.

The system we have given as a design example has been used to produce a self-checking calibrator for digital voltmeters. This calibrates the first decade of the lowest three ranges with an accuracy which is limited almost entirely by the stability and resolution of the voltmeter.

TABLE II
CALCULATED RATIOS FOR RESISTORS OFFSET BY ± 1 IN 10^3

Nominal ratio	Departure from nominal ppm			
	X	Y	$(XY)^{1/2}$	$1/2(X+Y)$
9:1	+ 667.8	− 665.6	+0.9	+1.1
8:2	+ 1.0	+ 1.0	+1.0	+1.0
7:3	+1048.5	− 1046.7	+0.4	+0.9
6:4	+1668.0	− 1665.3	0	+1.3
5:5	+2002.0	− 1998.0	0	+2.0

V. LIMITATIONS

It has been assumed in the analysis that the resistance of each of the necessary links is zero. In practice, it is difficult to reduce the resistance of a simple switch to less than about 1 mΩ and only in a few special cases can weighting resistors be used as in the Hamon buildup resistors. Hence, if an accuracy of 1 ppm is to be assured, the individual resistances must be at least 1 kΩ.

The relative values of the resistances must be stable during measurements with the two separate configurations of the ratio. So far as ambient temperature effects are concerned, the degree of equality of the temperature coefficients of the components is more important than the actual values. A limitation which must be borne in mind stems from the load coefficients of the resistors. Whether the ratio is being used as a potential or current divider each resistor will most likely be dissipating different powers in the two configurations.

VI. CONCLUSION

A technique has been described for setting up resistive ratios in such a way that the ratio may be obtained by two different configurations. *In situ* calibration of the ratio is made by taking two measurements, one for each configuration. Because of the need to switch the resistors, the technique is most useful for resistances of 1 kΩ or more. Errors due to the load coefficients of the resistors are not eliminated.

REFERENCES

[1] T. F. Finnegan and A. Denenstein, "High-accuracy potentiometers for use with ten millivolt Josephson devices," *Rev. Sci. Instrum.*, vol. 44, pp. 944–953, Aug. 1973.

[2] B. V. Hamon, "A 1–100 Ω build-up resistor for the calibration of standard resistors," *J. Sci. Instrum.*, vol. 31, pp. 450–453, Dec. 1954.