Analog Filter General Transfer Functions

This is a compendium containing the general and classic analog filter transfer functions in the complex (*s*) domain, aka Laplace notation.
I’ve compiled it because I’ve found no source for this kind of comprehensive listing anywhere else.
I personally find it very useful, and refer to it frequently when doing circuit analysis/synthesis.

Additionally, generic equations for calculating magnitude and phase response for second-order filters are included.

In General:

The complex operator s is used in all general transfer functions.
For analysis and synthesis in the real time/frequency domain, s is replaced by $j\omega$, where j indicates an imaginary number and ω the angular frequency.
For final analysis, $\omega=2\pi f$ where f is frequency in Hz.

Transfer functions higher than first order use the damping ratio ζ, but if you’re more comfortable using Q or α, the relationsip is: $2\zeta = 1/Q = \alpha$.

K is simply a gain factor and can be set to 1 at your choice, it’s just there for completeness.

Higher order transfer functions are listed as even/odd and can always be realized using cascaded second order filters, or second order filters plus an additional first order filter.

For the higher-order filters, everything after the ”…..” is optional.
Mostly a 2^{nd}, 3^{rd} or perhaps a 5^{th} order filter is useful. It’s all up to you.

Cheers.

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First-order filters:

Low pass:

$$F(s)=K\cdot\frac{\omega_0}{s+\omega_0}$$

High pass:

$$F(s)=K\cdot\frac{s}{s+\omega_0}$$

All pass:

$$F(s)=K\cdot\frac{s-\omega_0}{s+\omega_0}$$

Second- and higher-order filters:

Low pass even order:

$$F(s)=K\cdot\frac{\omega_{0a}^2}{s^2+2\,\zeta_a\,\omega_{0a}\,s+\omega_{0a}^2}\cdot\ldots\cdot\frac{\omega_{0n}^2}{s^2+2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}$$

Low pass odd order:

$$F(s)=K\cdot\frac{\omega_{0a}}{s+\omega_{0a}}\cdot\frac{\omega_{0b}^2}{s^2+2\,\zeta_b\,\omega_{0b}\,s+\omega_{0b}^2}\cdot\ldots\cdot\frac{\omega_{0n}^2}{s^2+2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}$$

High pass even order:

$$F(s)=K\cdot\frac{s^2}{s^2+2\,\zeta_a\,\omega_{0a}\,s+\omega_{0a}^2}\cdot\ldots\cdot\frac{s^2}{s^2+2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}$$

High pass odd order:

$$F(s)=K\cdot\frac{s}{s+\omega_{0a}}\cdot\frac{s^2}{s^2+2\,\zeta_b\,\omega_{0b}\,s+\omega_{0b}^2}\cdot\ldots\cdot\frac{s^2}{s^2+2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}$$

All pass even order:

$$F(s)=K\cdot\frac{s^2-2\,\zeta_a\,\omega_{0a}\,s+\omega_{0a}^2}{s^2+2\,\zeta_a\,\omega_{0a}\,s+\omega_{0a}^2}\cdot\ldots\cdot\frac{s^2-2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}{s^2+2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}$$

All pass odd order:

$$F(s)=K\cdot\frac{s-\omega_{0a}}{s+\omega_{0a}}\cdot\frac{s^2-2\,\zeta_b\,\omega_{0b}\,s+\omega_{0b}^2}{s^2+2\,\zeta_b\,\omega_{0b}\,s+\omega_{0b}^2}\cdot\ldots\cdot\frac{s^2-2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}{s^2+2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}$$

Band pass (peak) even order:

$$F(s)=K\cdot\frac{2\,\zeta_a\,\omega_{0a}\,s}{s^2+2\,\zeta_a\,\omega_{0a}\,s+\omega_{0a}^2}\cdot\ldots\cdot\frac{2\,\zeta_n\,\omega_{0n}\,s}{s^2+2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}$$

Band stop (notch) even order:

$$F(s)=K\cdot\frac{s^2+\omega_{0a}^2}{s^2+2\,\zeta_a\,\omega_{0a}\,s+\omega_{0a}^2}\cdot\ldots\cdot\frac{s^2+\omega_{0n}^2}{s^2+2\,\zeta_n\,\omega_{0n}\,s+\omega_{0n}^2}$$

Magnitude Response for Second-Order Filters

Determining the magnitude response $M(s)$ of a transfer function is done by calculating the absolute value of $F(s)$ by replacing s with $j\omega$ and solving for ω.

$$\boxed{M(s) = |F(s)| = \frac{|N(s)|}{|D(s)|}}$$

$N(s)$ and $D(s)$ are numerator and denominator of the transfer function $F(s)$.
Keep in mind that j denotes an imaginary number with a π/2 (or 90°) angle in the complex domain and that $j^2=-1$, so using $s=j\omega$ means $s^2=-\omega^2$.

<u>Example:</u> low pass filter transfer function, replacing s with $j\omega$:

$$F(s) = K \cdot \frac{\omega_0^2}{s^2 + 2\,\zeta\,\omega_0\,s + \omega_0^2} \qquad \text{gives:} \qquad F(j\,\omega) = K \cdot \frac{\omega_0^2}{-\omega^2 + 2\,\zeta\,\omega_0\,j\,\omega + \omega_0^2}$$

Rearranging the real and imaginary (j) terms and using only the absolute values for the vector lengths, by applying Pythagoras' theorem $\sqrt{(a^2 + b^2)} = c^2$, we get:

$$M(\omega) = K \cdot \left| \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\,\zeta\,\omega_0\,\omega)^2}} \right|$$

Taking this result further by factoring or expansion is not productive.
Important is, that the equation now has only one real variable (ω), and can be processed numerically in any graphing/plotting/analyser/spreadsheet environment.

The other transfer functions (high pass, band pass, band stop) are just as simple. The denominators are all the same, and the numerators are easy to calculate:

High pass:

$$M(\omega) = K \cdot \left| \frac{-\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\,\zeta\,\omega_0\,\omega)^2}} \right|$$

All pass:
magnitude is always 1, which is obvious by inspection.

Band pass:

$$M(\omega) = K \cdot \left| \frac{2\,\zeta\,\omega_0\,\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\,\zeta\,\omega_0\,\omega)^2}} \right|$$

Band stop:

$$M(\omega) = K \cdot \left| \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\,\zeta\,\omega_0\,\omega)^2}} \right|$$

Phase Response for Second-Order Filters

Finding the phase response is done by calculating $\Phi(s)$ which is the angular response of $F(s)$:

$$\boxed{\Phi(s) = arg[F(s)] = arg[\frac{N(s)}{D(s)}] = arg[N(s)] - arg[D(s)]}$$

Where $N(s)$ and $D(s)$ are numerator and denominator of $F(s)$.

This is traditionally done by using inverse tangens, (*arctan* or tan^{-1}), but this method is tedious, as it only provides results in the $-\pi/2 < \theta < \pi/2$ range, giving ambiguous results and needing further computation to find the actual angle/phase.
θ is the symbol for the resulting angle.

An easier approach is using the "two-argument arctangent" function, *arctan2*(x,*y*) or *atan2*(x,*y*), which delivers results in the $-\pi < \theta \leq \pi$ range as needed.
Arctan2 is not suitable for pen-and paper calculation, but is generally available in spreadsheet and mathematical programs.

The goal again is: find equations with only one real variable ω for further analysis.

Using *arctan2* instead of *arg*, replacing s with $j\omega$, and using *Re*[] and *Im*[] to extract the real and imaginary values, we get:

$$\Phi(\omega) = arctan2(Re[N(j\omega)], Im[N(j\omega)]) - arctan2(Re[D(j\omega)], Im[D(j\omega)])$$

($\Phi(\omega)$, as *Im*[] and *Re*[] eliminate j, leaving only real values).

Low pass:

$$\Phi(\omega) = arctan2(\omega_0^2, 0) - arctan2((\omega_0^2 - \omega^2), (2\zeta\omega_0\omega))$$

High pass:

$$\Phi(\omega) = arctan2(-\omega^2, 0) - arctan2((\omega_0^2 - \omega^2), (2\zeta\omega_0\omega))$$

All pass:

$$\Phi(\omega) = arctan2((\omega_0^2 - \omega^2), (-2\zeta\omega_0\omega)) - arctan2((\omega_0^2 - \omega^2), (2\zeta\omega_0\omega))$$

Band pass:

$$\Phi(\omega) = arctan2(0, 2\zeta\omega_0\omega) - arctan2((\omega_0{}^2 - \omega^2), (2\zeta\omega_0\omega))$$

Band stop:

$$\Phi(\omega) = arctan2((\omega_0{}^2 - \omega^2), 0) - arctan2((\omega_0{}^2 - \omega^2), (2\zeta\omega_0\omega))$$

Note that the denominator part is always the same, consistent with the *M(ω)* calculations.

Comments on Higher Order Filters:

The general equations (framed) for calculating magnitude and phase response shown in the previous section always apply, but the ones specific to second order filters only to those.
Simply multiplying first and second order magnitude responses will not work, nor will adding the phase responses.

For third, fourth, fifth etc. order filters, multiplication of their first and second order complex transfer functions is necessary to get 3rd, 4th, 5th etc. order polynomiums. Using those, the magnitude/phase calculation is possible and correct.

Why? Because each first or second order function usually have different ω_0 and ζ values.

Comments on Group Delay

Group delay is a function that expresses how much each signal frequency is delayed when passing through a system, in this case an analog filter. It is valuable when predicting signal integrity and step or impulse responses in the time domain.

Group delay is the derivative of phase response, meaning that a linear phase response over frequency equals a constant group delay.
The group delay function $\tau_g(\omega)$ is defined as:

$$\boxed{\tau_g(\omega) = -\frac{d\,\Phi(\omega)}{d\,\omega}}$$

Unfortunately, *arctan2(y,x)* does not lend itself easily to differentiation, which means numerical methods are indicated.