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A new four-dimensional hyperchaotic Lorenz system and its adaptive control

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Based on the Lorenz chaotic system, this paper constructs a new four-dimensional hyperchaotic Lorenz system, and studies the basic dynamic behaviours of the system. The Routh–Hurwitz theorem is applied to derive the stability conditions of the proposed system. Furthermore, based on Lyapunov stability theory, an adaptive controller is designed and the new four-dimensional hyperchaotic Lorenz system is controlled at equilibrium point. Numerical simulation results are presented to illustrate the effectiveness of this method.

Keywords: hyperchaotic Lorenz system, adaptive control, Lyapunov stability theory

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1. Introduction

Since Lorenz found the first chaotic attractor in the three-dimensional (3D) autonomous chaotic system in 1963,[1] people have realized that chaos is a ubiquitous and extremely complex nonlinear phenomenon in nature. From then on, chaos has been intensively studied in the last three decades.[2–5] Based on the Lorenz system, other chaotic systems were proposed in succession: Chen and Lü chaotic system[6] and Liu chaotic system.[7] These systems referred in these references are 3D chaotic system, each one has only one positive Lyapunov exponent. However, hyperchaotic system has two or more positive Lyapunov exponents, and the dynamic behaviours of hyperchaotic system are more complex and difficult to predict. In recent years, four-dimensional (4D) chaotic systems[8–11] are widely studied, and adaptive control method for 4D chaotic systems also has been concerned. Other control methods have been developed such as feedback control,[12–15] O–G–Y method,[16] sliding mode control,[17] impulsive control[18] and reduced-order adaptive control.[19]

In this paper, based on the Lorenz chaotic system and another 4D hyperchaotic Lorenz system,[8] a new 4D hyperchaotic Lorenz system is constructed and the basic dynamic behaviours are studied, such as bifurcation diagram, symmetry, equilibrium point and stability, Lyapunov dimension and the impact of system parameters. Furthermore, based on Lyapunov stability theory, an adaptive controller is designed and the new 4D hyperchaotic Lorenz system is controlled at equilibrium point via the adaptive control method. Numerical simulation results are presented to illustrate the effectiveness of this method.

2. A new 4D hyperchaotic Lorenz system

In Ref. [1], the Lorenz chaotic system is given by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - xz - y, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

where x, y, and z are state variables , $a = 10$, $b = 8/3$, and $c = 28$ are parameters.

As is well known, the 3D Lorenz chaotic system has only one positive Lyapunov exponent. However, hyperchaotic system must satisfy the following two necessary conditions: 1) for autonomous system, four-dimension (4D) is required at least; 2) two or more positive Lyapunov exponents and the sum of all the Lyapunov exponents is less than 0.

In Ref. [8], Wang *et al.* proposed a 4D hyperchaotic Lorenz system, and state equations are expressed as

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - y - xz + u, \\ \dot{z} = xy - bz, \\ \dot{u} = -kx. \end{cases} \tag{2}$$

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By referring the 4D hyperchaotic Lorenz system (2), which is proposed by Wang *et al.*, and making the following changes for the system (1), a new 4D hyperchaotic Lorenz system is constructed.

By increasing the fourth state variable w with parameter e in the second state equation of Lorenz system and the change rate of the fourth state variable w, furthermore changing nonlinear term in the third state equation of Lorenz system, state equations are expressed as

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - xz - y + ew, \\ \dot{z} = x^4 + y^4 - bz, \\ \dot{w} = -dy. \end{cases} \quad (3)$$

There are five parameters in this new 4D hyperchaotic Lorenz system (3), they are more than two parameters for the Lorenz system (1). The nonlinear term in the third state equation of system (3) is x^4+y^4, which is different from the systems (1) and (2). Four Lyapunov exponents of the new 4D hyperchaotic Lorenz system (3) are $\lambda_1 = 0.60613$, $\lambda_2 = 0.28066$, $\lambda_3 = 0$, and $\lambda_4 = -11.489$. The sum of all the Lyapunov exponents is less than 0. These results satisfy the above two necessary conditions.

Where x, y, z, and w are state variables and $a = 10$, $b = 8/3$, $c = 46$, $d = 2$, and $e = 12$ are parameters of the new 4D hyperchaotic Lorenz system (3). Hyperchaotic Lorenz attractors and phase plane strange attractors are shown in Fig. 1(a)–1(f); the bifurcation diagram of state variable x with the parameters c is shown in Fig. 2(a).

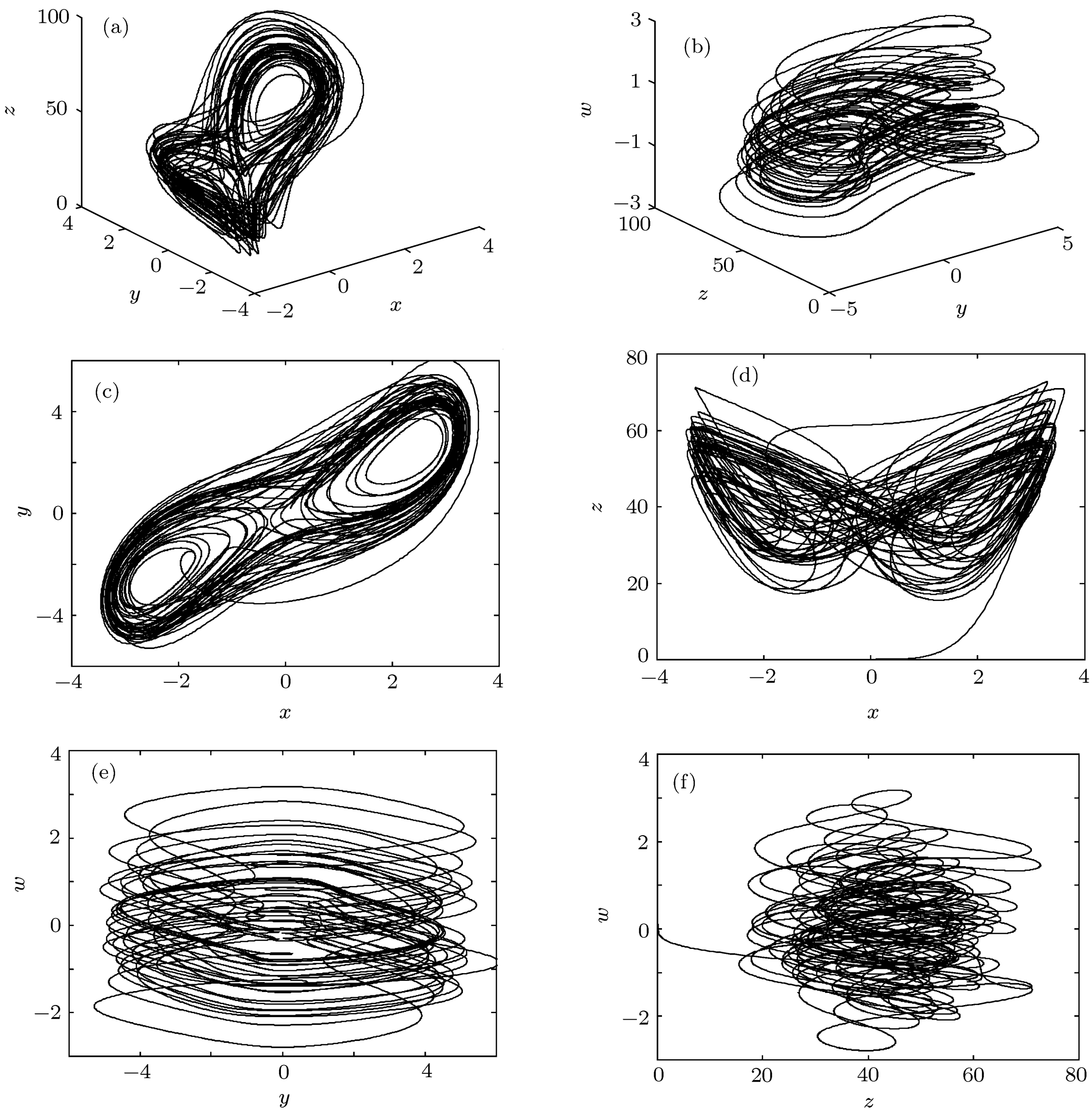


Fig. 1. Hyperchaotic attractors. (a) Hyperchaotic attractor in (x, y, z) space, (b) Hyperchaotic attractor in (y, z, w) space, (c) Phase diagram $(x - y)$, (d) Phase diagram $(x - z)$, (e) Phase diagram $(y - w)$, (f) Phase diagram $(z - w)$.

Only the local maximum and minimum of the state variable x are drawn in Fig. 2(a). A state trajectory of x is drawn in Fig. 2(b).

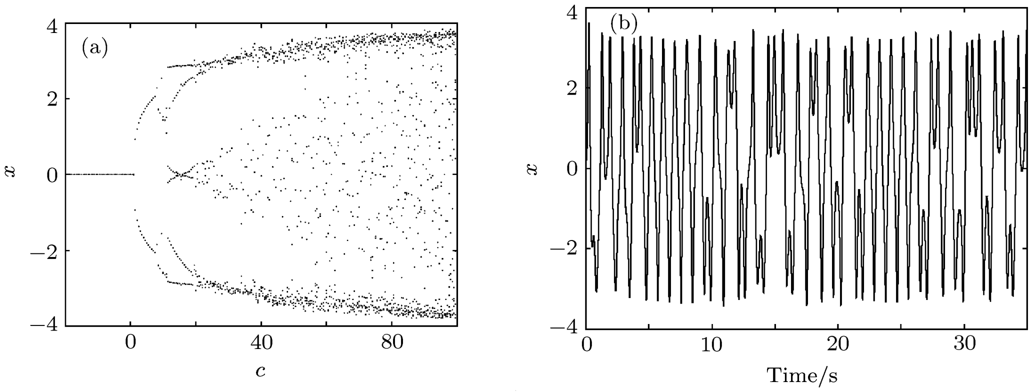


Fig. 2. (a) Bifurcation diagram of state variable x with the parameter c. (b) State trajectory of x.

3. Analysis of basic dynamic behaviour

3.1. Symmetry

Applying the coordinate transformation to the original system: $(x, y, z, w) \rightarrow (-x, -y, z, -w)$, if the equations of state remain unchanged, it means that the system's map is symmetric about the z-axis.

3.2. Dissipation and existence of attractor

The divergence of the system (3) is defined by

$$\begin{aligned}\nabla V &= \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} \\ &= -a - 1 - b = -\frac{41}{3}.\end{aligned} \tag{4}$$

Because $\nabla V < 0$, the system (3) is a dissipative system, and converges with an index rate of $\mathrm{e}^{-(41/3)t}$. Volume element V_0 shrinks to $V_0\mathrm{e}^{-(41/3)t}$ at the time t. When $t \rightarrow \infty$, volume element V_0 shrinks to 0. Therefore, all trajectories of the system will be confined to a congregation, whose volume is 0. And the gradual movement behaviours are fixed in an attractor.

3.3. Equilibrium point and stability

The equilibrium point of system (3) can be found by solving the following equations:

$$\begin{cases} a(y - x) = 0, \\ cx - xz - y + ew = 0, \\ x^4 + y^4 - bz = 0, \\ -dy = 0, \end{cases} \tag{5}$$

where only equilibrium point $O(0, 0, 0, 0)$ could be obtained.

For the given hyperchaotic Lorenz system (3), the Jacobian matrix at the equilibrium point $O(0, 0, 0, 0)$ is

$$J = \begin{bmatrix} -a & a & 0 & 0 \\ c & -1 & 0 & e \\ 0 & 0 & -b & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}. \tag{6}$$

Thus the corresponding characteristic equation can be obtained:

$$f(\lambda) = (\lambda + b)(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3) = 0, \tag{7}$$

where

$$\begin{cases} a_1 = a + 1, \\ a_2 = a + de - ac, \\ a_3 = ade. \end{cases} \tag{8}$$

The following equations can be got from Eq. (7):

$$\lambda_1 = -b, \tag{9}$$

and

$$f_1(\lambda) = \lambda^3 + (a+1)\lambda^2 + (a + de - ac)\lambda + ade = 0. \tag{10}$$

For $b > 0$, then $\lambda_1 < 0$. To make the system (3) to be stable at the equilibrium $O(0,0,0,0)$, the λ_2, λ_3, and λ_4 obtained from Eq. (10) must be less than 0. According to Routh–Hurwitz theorem, the system (3) being in stable equilibrium point $O(0,0,0,0)$ must satisfy the following conditions:

$$\begin{cases} b > 0, \\ a_1 = a + 1 > 0, \\ a_3 = ade > 0, \\ a_1 a_2 - a_3 = (a+1)(a + de - ac) - ade > 0. \end{cases} \tag{11}$$

For $a > 0$, $de > 0$ and $c < 1 + de/(a^2 + a)$ are obtained from the conditions (11), it means that the system (3) stabilizing the equilibrium point $O(0,0,0,0)$ must satisfy the following conditions:

$$\begin{cases} a, b, de > 0, \\ c < 1 + de/(a^2 + a). \end{cases} \tag{12}$$

For $a < 0$, $-1 < a < 0$, $de < 0$, and $c > 1 + de/(a^2 + a)$ are obtained from the conditions (11), it means that the system (3) stabilizing the equilibrium point $O(0,0,0,0)$ must satisfy the following conditions:

$$\begin{cases} -1 < a < 0, \\ b > 0, \\ de < 0, \\ c > 1 + de/(a^2 + a). \end{cases} \tag{13}$$

In this study, choosing parameters $a = 10$, $b = 8/3$, $d = 2$, and $e = 12$, we obtain $c < 1.21818$.

3.4. Calculating the Lyapunov dimension

As is well known, the Lyapunov exponents measure the exponential rates of divergence or convergence of nearby trajectories in phase space. The Lyapunov dimension of chaos attractors of this new hyperchaotic system is fractional, which is described as

$$\begin{aligned} D_L &= j + \frac{1}{|\lambda_{j+1}|} \sum_{i=1}^{j} \lambda_i \\ &= 3 + \frac{1}{|-11.489|}(0.60613 + 0.28066 + 0) \\ &= 3.077186. \end{aligned} \tag{14}$$

3.5. The impact of system parameters

In this study, parameters $a = 10$, $b = 8/3$, $d = 2$ and $e = 12$ were chosen, and by changing the value of c, the system (3) showed different attractors and movement forms as follows:

For $c = -5$, the of system (3) stability at the equilibrium point $O(0,0,0,0)$ as shown in Fig. 3.

For $c = 5$, the behaviours of the system (3) exhibit periodic motion, and result in the limit cycle as shown in Fig. 4.

For $c = 7.6$, the behaviours of the system (3) exhibit quasi-periodic motion, and result in the 2D torus as shown in Fig. 5.

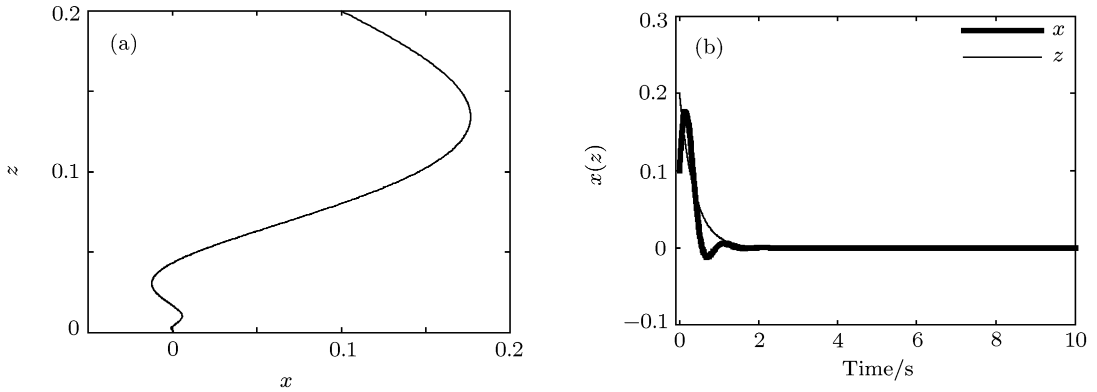


Fig. 3. Phase diagram and state trajectories. (a) Phase diagram $(x - z)$, (b) State trajectories of x and z.

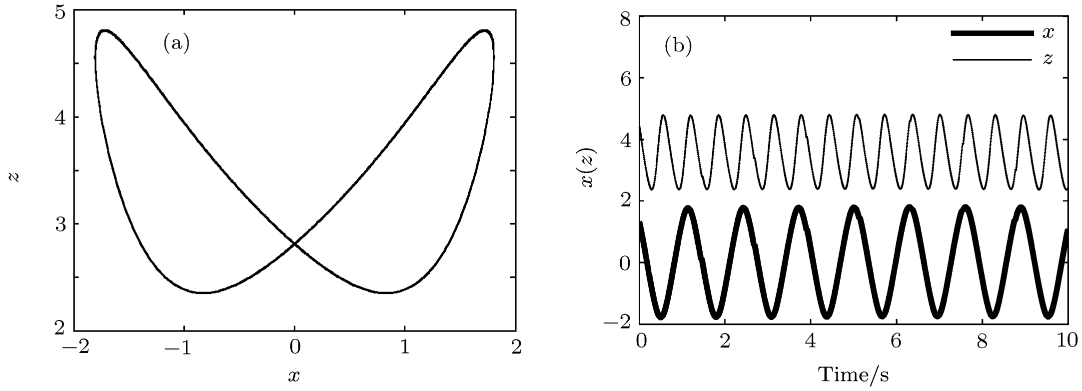


Fig. 4. Phase diagram and state trajectories. (a) Phase diagram $(x - z)$, (b) State trajectories of x and z.

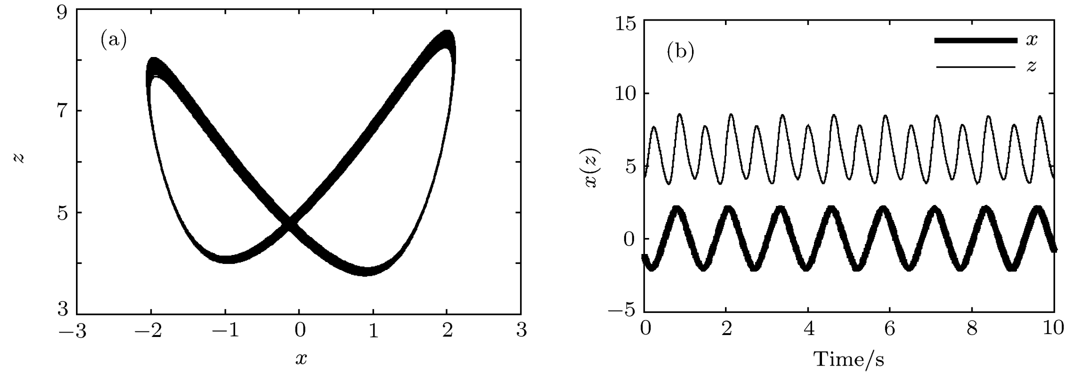


Fig. 5. Phase diagram and state trajectories. (a) Phase diagram $(x - z)$, (b) State trajectories of x and z.

For $c = 9$, the behaviours of the system (3) also exhibit periodic motion, and result in another limit cycle as shown in Fig. 6.

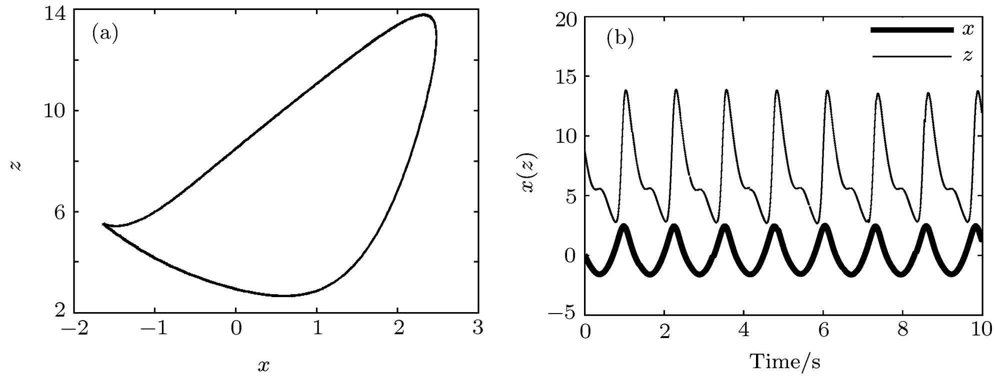


Fig. 6. Phase diagram and state trajectories. (a) Phase diagram $(x - z)$, (b) State trajectories of x and z.

For $c = 14$, the behaviours of the system (3) exhibit periodic motion, and result in the complex limit cycle as shown in Fig. 7.

For $c = 13$, the behaviours of the system (3) exhibit chaotic motion, and result in the chaotic attractor as shown in Fig. 8.

For $c = 21$, the behaviours of the system (3) exhibit hyperchaotic motion, and result in the hyperchaotic attractor as shown in Fig. 9.

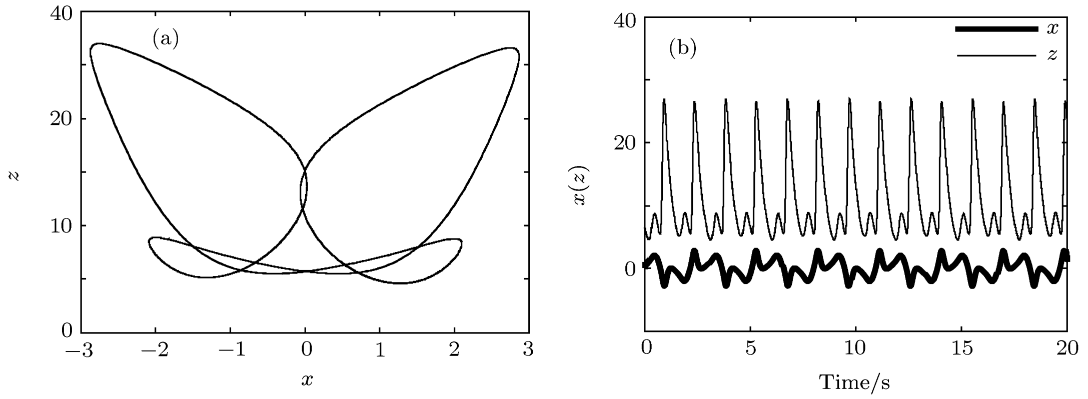


Fig. 7. Phase diagram and state trajectories. (a) Phase diagram $(x - z)$, (b) State trajectories of x and z.

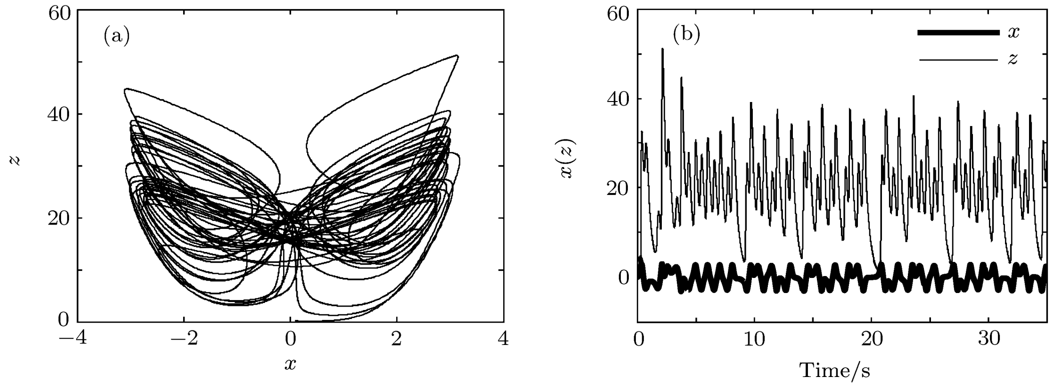


Fig. 8. Phase diagram and state trajectories. (a) Phase diagram $(x - z)$, (b) State trajectories of x and z.

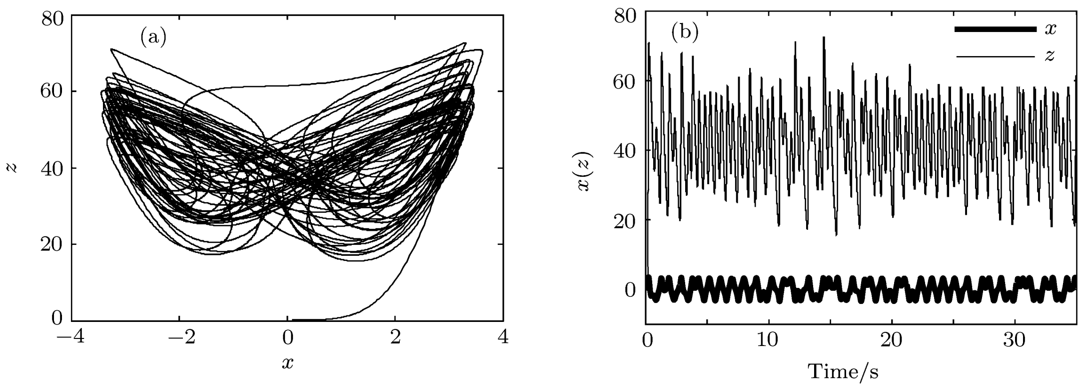


Fig. 9. Phase diagram and state trajectories. (a) Phase diagram $(x - z)$, (b) State trajectories of x and z.

4. Adaptive control of the new 4D hyperchaotic Lorenz system

In this section, our goal is to design an adaptive controller $u(t) = (u_1, u_2, u_3, u_4)^{\mathrm{T}}$, and the adaptive parameters $k = (k_1, k_2, k_3, k_4)^{\mathrm{T}}$ of the controller $u(t)$ can automatically adjust the controller $u(t)$. Finally, the new 4D hyperchaotic Lorenz system (3) is controlled at equilibrium point $O(0, 0, 0, 0)$ under the adaptive controller $u(t)$.

The new 4D hyperchaotic Lorenz system including the control input $u(t) = (u_1, u_2, u_3, u_4)^{\mathrm{T}}$ is expressed as

$$\begin{cases} \dot{x} = a(y-x) + u_1, \\ \dot{y} = cx - xz - y + ew + u_2, \\ \dot{z} = x^4 + y^4 - bz + u_3, \\ \dot{w} = -dy + u_4. \end{cases} \tag{15}$$

where x, y, z, and w are state variables, $a = 10$, $b = 8/3$, $c = 46$, $d = 2$, and $e = 12$ are parameters of the system (15).

In this study, the adaptive controller is designed as

$$\begin{cases} u_1 = k_1 x, \\ u_2 = xz + k_2 y, \\ u_3 = -(x^4 + y^4) + k_3 z, \\ u_4 = k_4 w, \end{cases} \tag{16}$$

where k_1, k_2, k_3, and k_4 are adaptive parameters of the control input $u(t)$.

The adaptive laws of adaptive parameters k_1, k_2, k_3, and k_4 are designed as

$$\begin{cases} \dot{k}_1 = -\alpha x^2, \\ \dot{k}_2 = -\alpha y^2, \\ \dot{k}_3 = -\alpha z^2, \\ \dot{k}_4 = -\alpha w^2, \end{cases} \tag{17}$$

where α is a constant larger than zero, and it can control the speed of convergence of adaptive laws.

The Lyapunov function for the system (15) is chosen as

$$\begin{aligned} V(t) \;=\; & \frac{1}{2}(x^2 + y^2 + z^2 + w^2) \\ & + \frac{1}{2\alpha}[(k_1 - a + \lambda)^2 + (k_2 - 1 + \lambda)^2 \\ & + (k_3 - b + \lambda)^2 + (k_4 + \lambda)^2] > 0. \end{aligned} \tag{18}$$

where λ is a constant larger than zero.

Then, taking the derivative of Eq. (18) with respect to time, we have

$$\dot{V}(t) = x\dot{x} + y\dot{y} + z\dot{z} + w\dot{w} + \frac{1}{\alpha}[(k_1 - a + \lambda)\dot{k}_1 + (k_2 - 1 + \lambda)\dot{k}_2 + (k_3 - b + \lambda)\dot{k}_3 + (k_4 + \lambda)\dot{k}_4]. \tag{19}$$

Substituting the system (15) , Eq. (16) and Eq. (17) into Eq. (19), we can express $\dot{V}(t)$ as

$$\begin{aligned} \dot{V}(t) \;=\; & x[a(y-x) + u_1] + y(cx - xz - y + ew + u_2) + z(x^4 + y^4 - bz + u_3) + w(-dy + u_4) \\ & + \frac{1}{\alpha}[(k_1 - a + \lambda)(-\alpha x^2) + (k_2 - 1 + \lambda)(-\alpha y^2) + (k_3 - b + \lambda)(-\alpha z^2) + (k_4 + \lambda)(-\alpha w^2)] \\ =\; & x[a(y-x) + k_1 x] + y(cx - xz - y + ew + xz + k_2 y) + z(x^4 + y^4 - bz - (x^4 + y^4) + k_3 z) + w(-dy + k_4 w) \\ & + [(k_1 - a + \lambda)(-x^2) + (k_2 - 1 + \lambda)(-y^2) + (k_3 - b + \lambda)(-z^2) + (k_4 + \lambda)(-w^2)] \\ =\; & x[a(y-x) + k_1 x] + y(cx - y + ew + k_2 y) + z(-bz + k_3 z) + w(-dy + k_4 w) \\ & + [(k_1 - a + \lambda)(-x^2) + (k_2 - 1 + \lambda)(-y^2) + (k_3 - b + \lambda)(-z^2) + (k_4 + \lambda)(-w^2)] \\ =\; & -\lambda x^2 + (a + c)xy - \lambda y^2 - \lambda z^2 + (e - d)yw - \lambda w^2 \\ =\; & -[x, y, z, w] \begin{bmatrix} \lambda & -(a+c) & 0 & 0 \\ 0 & \lambda & 0 & d-e \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = -X^{\mathrm{T}} \boldsymbol{P} X, \end{aligned} \tag{20}$$

where $X = [x, y, z, w]^{\mathrm{T}}$ is state vector and $P = \begin{bmatrix} \lambda & -(a+c) & 0 & 0 \\ 0 & \lambda & 0 & d-e \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$. Because the matrix $\boldsymbol{P}$ is a positive definite matrix, $\dot{V}(t) < 0$. Finally, the new 4D hyperchaotic Lorenz system (3) is controlled at equilibrium point $O(0,0,0,0)$.

From the above results of the analysis, as long as we give the initial values of the adaptive parameters k_1, k_2, k_3, and k_4, these parameters can automatically adjust the controller $u(t)$ based on the adaptive controller $u(t)$, which is designed in this study. Finally, the new 4D hyperchaotic Lorenz system (3) is controlled at equilibrium point $O(0,0,0,0)$.

5. Simulation results

In this section, the new 4D hyperchaotic Lorenz system (3) is controlled at equilibrium point via adaptive control method. The step size in the simulation is chosen as 0.001 s. The initial values of the new 4D hyperchaotic Lorenz system (3) and adaptive parameters are $(x_1(0), x_2(0), x_3(0), x_4(0)) = (10, -15, 5, -5)$ and $(k_1(0), k_2(0), k_3(0), k_4(0)) = (5, 10, -5, -10)$, respectively.

Case 1 choose $\alpha = 1$ in Eq. (17), the new 4D hyperchaotic Lorenz system (3) is controlled at equilibrium point via adaptive control method. Numerical simulation results are shown in Fig. 10.

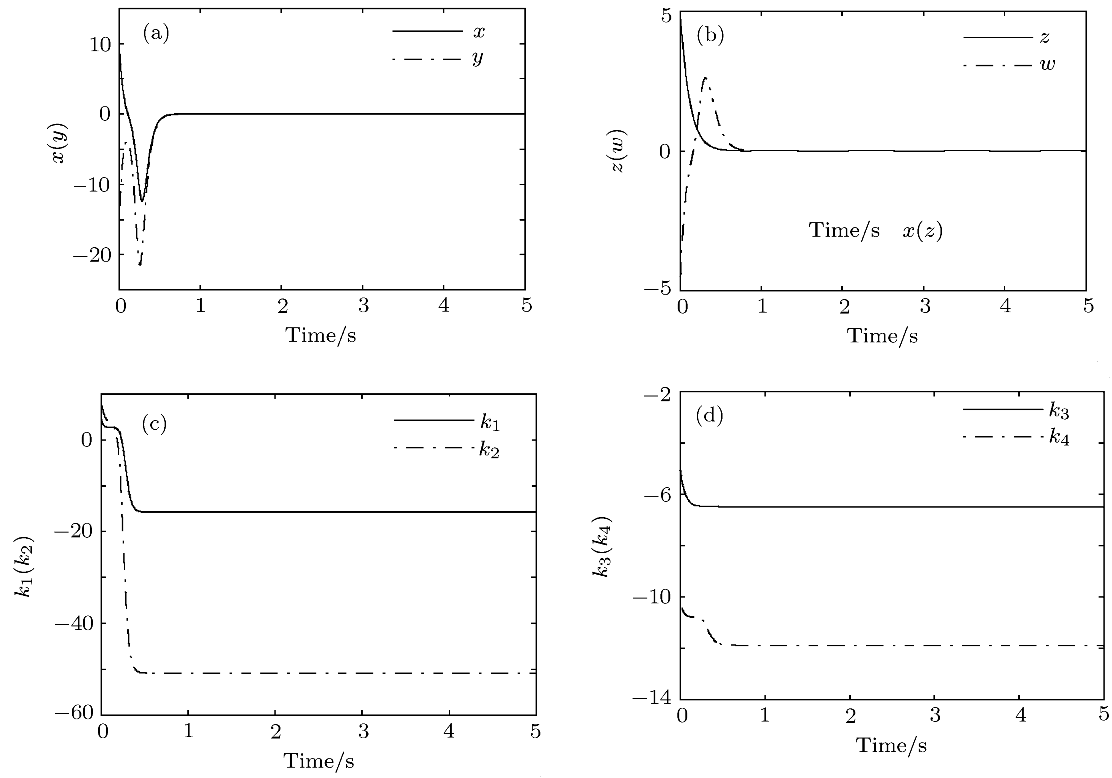
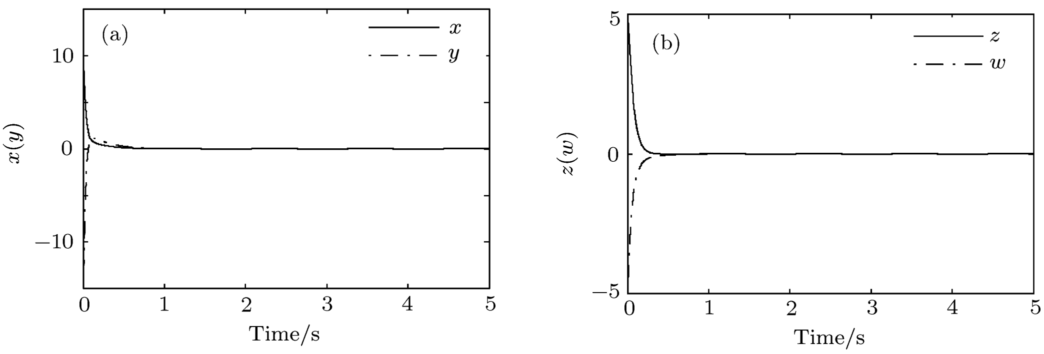


Fig. 10. (a) Time waveforms of state trajectories x (solid) and y (dashed). (b) Time waveforms of state trajectories z (solid) and w (dashed). (c) Variation of adaptive parameters k_1 (solid) and k_2 (dashed) with time t. (d) Variation of adaptive parameters k_3 (solid) and k_4 (dashed) with time t.

Case 2 choose $\alpha = 10$ in Eq. (17), the new 4D hyperchaotic Lorenz system (3) is controlled at equilibrium point via adaptive control method. Numerical simulation results are shown in Fig. 11.

The above numerical simulation results are presented to illustrate the effectiveness of this adaptive control method. Furthermore, α not only change the speed of convergence of adaptive laws, but also change the speed of convergence of state variables. Finally, the new 4D hyperchaotic Lorenz system can be controlled at equilibrium point by using the adaptive control method.



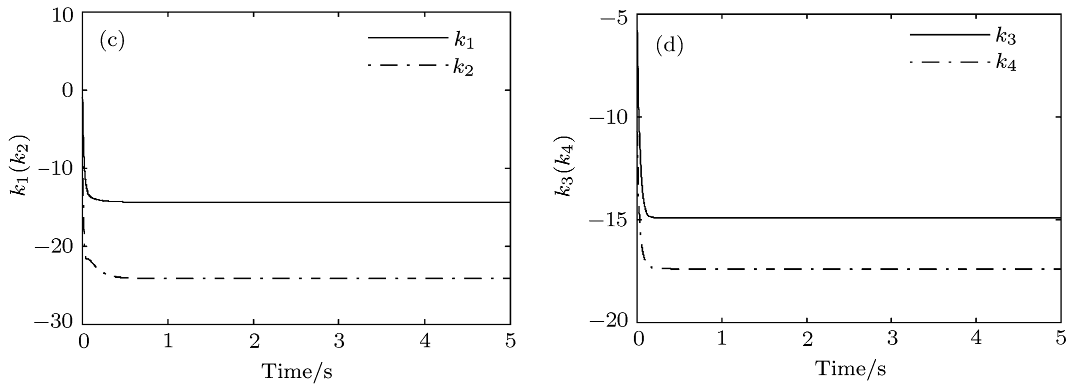


Fig. 11. (a) Time waveforms of state trajectories x (solid) and y (dashed). (b) Time waveforms of state trajectories z (solid) and w (dashed). (c) Variation of adaptive parameters k_1 (solid) and k_2 (dashed) with time t. (d) Variation of adaptive parameters k_3 (solid) and k_4 (dashed) with time t.

6. Conclusions

First, based on the Lorenz system and another 4D hyperchaotic Lorenz system,[8] increasing the state variables and changing nonlinear terms and parameters of the Lorenz system, we construct a new 4D hyperchaotic Lorenz system. Second, researches on the basic dynamic behaviours of the new 4D hyperchaotic Lorenz system reveal that the new 4D hyperchaotic Lorenz system has abundant dynamic behaviours. Finally, an adaptive controller is designed based on Lyapunov stability theory. Numerical simulation results show that the new 4D hyperchaotic Lorenz system can be controlled at equilibrium point via the adaptive control method.

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